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THERMOELASTIC INTERACTION OF HOT SPOTS WITH THE NONASYMPTOTIC EDGE OF A PHYSICALLY ORTHOTROPIC SHELL

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We study a circular cylindrical shell of a physically orthotropic material which has an arbitrary number of periodically located hot spots along some contour. In a semi-infinite shell, hot spots can be located in the immediate vicinity of the nonasymptotic edge of the shell, and, in some particular cases, even neighbor it. In the adopted formulation, a hot spot can be treated as a foreign inclusion whose coefficient of linear expansion differs from that of the main material of the shell at a constant temperature of the shell. The coefficients of linear expansion in the longitudinal and circumferential directions are assumed to be different; the product of the coefficient of linear expansion and the elastic modulus is constant over the shell. Such studies are of both theoretical and practical interest in connection with the problem of the thermal strength of structural elements of various high-temperature installations made of high-melting materials (for example, zirconium carbides) with the occurrence in them of temperature fields of large degree of localization — hot spots. Precisely in these zones, in which the localized stress state due to the occurrence of hot spots or macroinclusions is imposed on the overall stress state, microcrack initiation is possible. In brittle materials, this can lead to failure even under a single action of hot spots. If the material has sufficient plastic properties, failure occurs under repeated actions or thermal cycling.

Using full equations of the theory of physically orthotropic, elastic, thin shells based on the Kirchhoff–Love hypothesis, one can reduce the problem of the action of a temperature field $t(\alpha,\beta)$ on a shell to the following two differential equations for the resolving functions $\Phi^*(\alpha,\beta)$ and $\Phi^{**}(\alpha,\beta)$ [1]:

$$\mathcal{L}\Phi^*(\alpha,\beta) = (\alpha_{2t}+\nu_1\alpha_{1t})c^{-2}Rt^*(\alpha,\beta), \quad \mathcal{L}\Phi^{**}(\alpha,\beta) = -(6c^2)^{-1}(\alpha_{2t}+\nu_1\alpha_{1t})ht^{**}(\alpha,\beta). \tag{1}$$

Here

$$\mathcal{L} = \frac{\partial^8}{\partial\alpha^8} + a_{6,2}\frac{\partial^8}{\partial\alpha^6\partial\beta^2} + 2\nu_2\frac{\partial^6}{\partial\alpha^6} + a_{4,4}\frac{\partial^8}{\partial\alpha^4\partial\beta^4} + a_{4,2}\frac{\partial^6}{\partial\alpha^4\partial\beta^2} + \lambda\frac{\partial^4}{\partial\alpha^4} +$$
$$+ a_{2,6}\frac{\partial^8}{\partial\alpha^2\partial\beta^6} + a_{2,4}\frac{\partial^6}{\partial\alpha^2\partial\beta^4} + a_{2,2}\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \lambda^2\frac{\partial^4}{\partial\beta^4}\Big(\frac{\partial^2}{\partial\beta^2}+1\Big)^2 + \frac{1-\nu_1\nu_2}{c^2}\lambda\frac{\partial^4}{\partial\alpha^4};$$

$$a_{6,2} = \frac{\lambda-\nu_2^2}{\mu_1} + 4\mu_1; \quad a_{4,4} = 2\lambda\Big[3 + \frac{\nu_1}{\mu_2}(1-\nu_1\nu_2) - 4\nu_1(\nu_2+\mu_1)\Big]; \quad a_{4,2} = a_{4,4};$$

$$a_{2,6} = \lambda a_{6,2}; \quad a_{2,4} = 2\lambda(a_{6,2}-\nu_2); \quad \mu_2 = (G/E_2)(1-\nu_1\nu_2); \quad a_{2,2} = \lambda((\lambda-\nu_2^2)/\mu_1 - 2\nu_2);$$

$$\mu_1 = (G/E_1)(1-\nu_1\nu_2); \quad \lambda = E_2/E_1 = \nu_2/\nu_1; \quad c^2 = h^2/12R^2,$$

where α and β are nondimensional longitudinal and circumferential coordinates, R and h are the radius and thickness of the shell, E_1 and E_2 are the elastic moduli of the shell material in the α and β directions, respectively, G is the shear modulus, ν_1 is the coefficient of transverse compression in the β direction with extension in the α direction, ν_2 is the coefficient of transverse compression in the α direction with extension in the β direction, α_{1t} and α_{2t} are the coefficients of linear thermal expansion in the α and β directions,

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respectively, $t^*(\alpha,\beta) = (t_1 + t_2)/2$, $t^{**}(\alpha,\beta) = (t_2 - t_1)/2$, and $t_2(\alpha,\beta)$ and $t_1(\alpha,\beta)$ are the temperatures of the inner and outer surfaces of the shell.

Displacements, forces, bending moments, and other factors are related to the resolving functions $\Phi^*(\alpha,\beta)$ and $\Phi^{**}(\alpha,\beta)$ by differential relations [2].

In what follows, we shall study only the action of the temperature field $t^*(\alpha,\beta)$ on the shell, bearing in mind that the solution for the temperature field $t^{**}(\alpha,\beta)$ can be constructed in a similar manner (the superscript asterisk is omitted below).

Numerical solution of the equations of the general theory of cylindrical orthotropic shells involves definite difficulties. The high order of the resolving equations (1) and the cumbersome expressions of the desired factors in terms of the resolving function practically do not permit one to obtain convenient analytic solutions of the boundary-value problems. Moreover, solutions cannot be written in closed form or as explicit formulas without series. Therefore, in solving the boundary-value problems, we invoke methods of asymptotic synthesis (MAS) of stress–strain states [5, 2]. For this, we write approximate equations of semimomentless theory and simple edge effect, generalized Vlasov–Donnel equations, and equations for the tangential state.

The equation of semimomentless theory is obtained by simplifying the first of the resolving equations (1) using the strong inequality $(|\partial^2\Phi/\partial\beta^2|) \gg (|\partial^2\Phi/\partial\alpha^2|)$:

$$\frac{\partial^4\Phi}{\partial\alpha^4} + \frac{\lambda c^2}{1-\nu_1\nu_2}\,\frac{\partial^4}{\partial\beta^4}\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\Phi = \frac{\alpha_{2t}+\nu_1\alpha_{1t}}{\lambda(1-\nu_1\nu_2)}\,Rt(\alpha,\beta). \tag{2}$$

The resolving equation (2), obtained from (1) by means of Novozhilov's criterion [3], is a generalization of the Vlasov resolving equation [4] of semimomentless theory and describes here the basic state.

The resolving equation of simple nonaxisymmetric edge effect

$$\frac{\partial^4 w}{\partial\alpha^4} + \lambda\,\frac{1-\nu_1\nu_2}{c^2}\,w = (\nu_2\vartheta - \lambda)\,\frac{\alpha_{2t}+\nu_1\alpha_{1t}}{c^2}\,Rt(\alpha,\beta) \tag{3}$$

is obtained from (1) by using the strong inequalities

$$\left|\frac{\partial^2\Phi}{\partial\beta^2}\right| \ll \left|\frac{\partial^2\Phi}{\partial\alpha^2}\right|, \qquad \left|\frac{\partial^2\Phi}{\partial\alpha^2}\right| \gg |\Phi|.$$

The resolving equation of shallow shell theory (the Vlasov–Donnel equation) extended to physically orthotropic shells is of the form

$$\left(\frac{\partial^8}{\partial\alpha^8}+a_{6,2}\frac{\partial^8}{\partial\alpha^6\partial\beta^2}+a_{4,4}\frac{\partial^8}{\partial\alpha^4\partial\beta^4}+a_{2,6}\frac{\partial^8}{\partial\alpha^2\partial\beta^6}+\lambda^2\frac{\partial^8}{\partial\beta^8}+\lambda\frac{1-\nu_1\nu_2}{c^2}\frac{\partial^4}{\partial\alpha^4}\right)\Phi(\alpha,\beta) = \frac{\alpha_{2t}+\nu_1\alpha_{1t}}{c^2}Rt(\alpha,\beta). \tag{4}$$

Equation (4) is obtained from (1) by retaining only higher-order derivatives, i.e., eighth-order derivatives.

When the variability of the stress state is very high (the estimate is given below), the last term in Eq. (4) can be ignored. Equation (4) then becomes a polyharmonic equation, and the stress–strain state splits into two independent states: tangential and bending. The first of them is similar to the plane elastic problem and occurs under loading of the centroidal surface of a shell and under the temperature field $t^*(\alpha,\beta)$, which is constant across the thickness. The second is similar to the bending of a plate and occurs under transverse loading and the temperature field $t^{**}(\alpha,\beta)$.

When the temperature field is constant across the shell thickness, for the tangential state we obtain the resolving equation

$$\frac{\partial^4\varphi}{\partial\alpha^4} + \left(\frac{\lambda-\nu_2^2}{\mu_1} - 2\nu_2\right)\frac{\partial^4\varphi}{\partial\alpha^2\partial\beta^2} + \lambda\,\frac{\partial^4\varphi}{\partial\beta^4} = (\alpha_{2t}+\nu_1\alpha_{1t})\,Rt(\alpha,\beta). \tag{5}$$

We consider semi-infinite shells with a free edge and with a rigidly clamped edge under the action of a system of hot spots that occur at the free (rigid) edge of the shell. We place the coordinate origin at the center of one heated region (at the center of a hot spot) and represent the temperature field in the form of a Fourier series in the circumferential direction and in the form of the Fourier integral in the longitudinal direction:

$$t(\alpha,\beta) = t_0\theta(\alpha)\sum_{n=0}^{\infty}\theta_n\cos kn\beta, \qquad \theta(\alpha) = \frac{2}{\pi}\int_0^{\infty}\cos\alpha\omega\,d\omega\int_0^{\infty}\theta(\alpha)\cos\omega\alpha\,d\alpha. \tag{6}$$

Here t_0 is an amplitude temperature value, $\theta(\alpha)$ is a nondimensional function of temperature distribution along the generatrix, θ_n is a coefficient of the Fourier series, and k is the number of hot spots in the fixed section of the shell $\alpha = 0$.

We seek a solution of the resolving equation (1) in the form

$$\Phi(\alpha,\beta) = \sum_{n=0}^{\infty}\Phi_n(\alpha)\cos kn\beta. \tag{7}$$

Substituting (6) and (7) into (1), we obtain the following ordinary differential equation for the function $\Phi_n(\alpha)$:

$$\bar{\mathcal{L}}\Phi_n(\alpha) = (\alpha_{2t} + \nu_1\alpha_{1t})c^{-2}Rt_0\theta_n\theta(c); \tag{8}$$

$$\bar{\mathcal{L}} = \frac{d^8}{d\alpha^8} + (2\nu_2 - a_{6,2}k^2n^2)\frac{d^6}{d\alpha^6} + (a_{4,4}k^4n^4 - a_{4,2}k^2n^2 + \lambda)\frac{d^4}{d\alpha^4}$$
$$-(a_{2,6}k^6n^6 - a_{2,4}k^4n^4 + a_{2,2}k^2n^2)\frac{d^2}{d\alpha^2} + \lambda^2k^4n^4(k^2n^2-1)^2 + (1-\nu_1\nu_2)c^{-2}\lambda\frac{d^4}{d\alpha^4}. \tag{9}$$

For the shells considered, the solution must satisfy the boundary conditions at infinity and the following boundary conditions at the shell edge for $\alpha = -\xi$:

— the free edge

$$T_{1n}(-\xi) = S_n(-\xi) = 0 \qquad \text{(tangential conditions)}; \tag{10}$$

$$Q^*_{1n}(-\xi) = G_{1n}(-\xi) = 0 \qquad \text{(nontangential conditions)}; \tag{11}$$

— the clamped edge

$$u_n(-\xi) = v_n(-\xi) = 0 \qquad \text{(tangential conditions)}; \tag{12}$$

$$w_n(-\xi) = w'_{\alpha n}(-\xi) = 0 \qquad \text{(nontangential conditions)}. \tag{13}$$

Numerical realization of boundary-value problems for the differential equation (8) with the boundary conditions (10)–(13) involves, as noted above, some difficulties. They can be overcome by using the MAS of a stress state, formulated in [5] and developed in [2]. MAS allow one to replace the boundary-value problem for the differential equation (8) with the boundary conditions (10)–(13) by a set of boundary-value problems for differential equations of simpler structure and lower order: (2)–(5). In [5, 2], three MAS were proposed:

(1) the first uses the condition of minimum of the asymptotic error and equations of semimomentless theory and edge effect at "low" harmonic numbers n, and Vlasov–Donnel equations for "medium" and "high" n;

(2) the second uses equations of semimomentless theory and edge effect for "low" n, Vlasov–Donnel equations for "medium" n, and equations of the plane elastic problem and plate bending for "high" n;

(3) the third method uses equations of semimomentless theory and edge effect for "low" and "medium" n and equations of the type of the plane problem and plate bending for "high" n.

The "low," "medium," and "high" harmonic numbers include, respectively $n \leqslant \bar{n}$, $\bar{n}+1 \leqslant n \leqslant n^*$, and $n \geqslant n^* + 1$. For physically orthotropic shells, the formulas for the harmonic numbers $n = \bar{n}$ and $n = n^*$, which determine the limits of application of the approximate equations (2)–(5), have the form

$$\tilde{n}^4 \approx 2\sqrt{3}\frac{R}{h}\left[\sqrt{\lambda}\left(\frac{1-\nu_1\nu_2}{2\sqrt{\mu_1\mu_2}} + 2\sqrt{\mu_1\mu_2}\right) - \frac{\nu_2}{2}\right]\left(\frac{1-\nu_1\nu_2}{2\sqrt{\mu_1\mu_2}} + 2\sqrt{\mu_1\mu_2}\right)^{-2},$$
$$\tilde{n}^4 \approx 6(1-\nu_1\nu_2)(R/h)^2\sqrt{R/h}[3 + (\nu_1/\mu_2)(1-\nu_1\nu_2) - 4\nu_1(\nu_2+\mu_1)]^{-1}, \quad \tilde{n} = kn. \tag{14}$$

For the harmonic number n, the first of formulas (14) gives the value $\bar{n}$, rounded to the nearest integer, and the second of these formulas gives the value n^*.

To solve the formulated boundary-value problems, we use the second MAS, which leads to solution of the equations of semimomentless theory and edge effect (2) and (3) for $n \leqslant \bar{n}$, the equations of shallow shell theory (4) for $\bar{n}+1 \leqslant n \leqslant n^*$, and the equations of the tangential state (5) for $n \geqslant n^*+1$. Then, according to this MAS, the resolving function can be represented approximately as

$$\Phi_n(\alpha) \approx \Phi_n^{\mathrm{b}}(\alpha) + \Phi_n^{\mathrm{e}}(\alpha) \ \ (n \leqslant \bar{n}), \quad \Phi_n(\alpha) \approx \Phi_n^{\mathrm{s}}(\alpha) \ \ (\bar{n}+1 \leqslant n \leqslant n^*),$$
$$\Phi_n(\alpha) \approx \Phi_n^{\mathrm{t}}(\alpha) \quad (n \geqslant n^*+1). \tag{15}$$

Here the functions $\Phi_n^{\mathrm{b}}(\alpha)$, $\Phi_n^{\mathrm{e}}(\alpha)$, $\Phi_n^{\mathrm{s}}(\alpha)$, and $\Phi_n^{\mathrm{t}}(\alpha)$ describe the basic state, edge effect, the stress state based on shallow shell theory, and the tangential state (superscripts b, e, s, and t, respectively).

Instead of (8) for the function $\Phi_n^{\mathrm{b}}(\alpha)$, using (2) we obtain the equation

$$\Big(\frac{d^4}{d\alpha^4} + 4\mu_n^4\Big)\Phi_n^{\mathrm{b}}(\alpha) = (\alpha_{2t} + \nu_1\alpha_{1t})\lambda^{-1}(1-\nu_1\nu_2)^{-1}Rt_0\theta_n\theta(\alpha)$$
$$(4\mu_n^4 = c^2\lambda(1-\nu_1\nu_2)^{-1}\tilde{n}^4(\tilde{n}^2-1)^2, \quad \tilde{n} = kn). \tag{16}$$

The solution of this equation is the sum of the solution of the homogenous equation ($t_0 = 0$) and the particular solution found by applying a Fourier transform to (16):

$$\Phi_n^{\mathrm{b}}(\alpha) = c_1^{\mathrm{b}}\chi_n(\alpha) + c_2^{\mathrm{b}}\zeta_n(\alpha) + (2\pi\lambda)^{-1}(1-\nu_1\nu_2)^{-1}(\alpha_{2t}+\nu_1\alpha_{1t})Rt_0\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{-i\omega\alpha}}{\omega^4+4\mu_n^4}\,d\omega\int\limits_{-\infty}^{+\infty}\theta(\alpha)\mathrm{e}^{i\omega\alpha}d\alpha. \tag{17}$$

Here the functions $\chi_n(\alpha)$ and $\zeta_n(\alpha)$ are given by the formulas $\chi_n(\alpha) = \exp(-\mu_n\alpha)\cos\mu_n\alpha$, $\zeta_n(\alpha) = \exp(-\mu_n\alpha)\sin\mu_n\alpha$, and the arbitrary constants c_1^{b} and c_2^{b} are determined from the boundary conditions (10) for the free edge of the shell or from the boundary conditions (12) for the clamped edge of the shell.

For the resolving function of the edge effect, which is assumed to be radial displacement, we have the differential equation

$$\Big(\frac{d^4}{d\alpha^4} + 4\eta^4\Big)w_n^{\mathrm{e}}(\alpha) = -(\lambda - \nu_2\vartheta)(\alpha_{2t}+\nu_1\alpha_{1t})\,c^{-2}Rt_0\theta_n\theta(\alpha), \tag{18}$$

where $\vartheta = (\alpha_{1t}+\nu_2\alpha_{2t})/(\alpha_{2t}+\nu_1\alpha_{1t})$ and $4\eta^4 = \lambda(1-\nu_1\nu_2)\,c^{-2}$.

The solution of Eq. (18) with allowance for the boundary conditions at infinity is written as

$$w_n^{\mathrm{e}}(\alpha) = c_1^{\mathrm{e}}\chi(\alpha) + c_2^{\mathrm{e}}\zeta(\alpha) - (2\pi c^2)^{-1}(\lambda-\nu_2\vartheta)(\alpha_{2t}+\nu_1\alpha_{1t})Rt_0\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{-i\omega\alpha}}{\omega^4+4\eta^4}\,d\omega\int\limits_{-\infty}^{+\infty}\theta(\alpha)\mathrm{e}^{i\omega\alpha}d\alpha. \tag{19}$$

The functions $\chi(\alpha)$ and $\zeta(\alpha)$ are given by the formulas $\chi(\alpha) = \exp(-\eta\alpha)\cos\eta\alpha$ and $\zeta(\alpha) = \exp(-\eta\alpha)\sin\eta\alpha$, and the arbitrary constants c_1^{e} and c_2^{e} are found from the boundary conditions (11) or (13) for the free and rigidly clamped edges, respectively.

The mismatch in the boundary conditions, which appears as a result of separate imposition of the tangential and nontangential boundary conditions, is eliminated by means of a correcting edge effect for $\alpha = -\xi$ ($\alpha^+ = \alpha + \xi = 0$) using the relations

$$Ehw_n^{\mathrm{e}}(\alpha^+,\beta) = 2\eta[c_1\chi(\alpha^+) - c_2\zeta(\alpha^+)], \quad RT_{2n}^{\mathrm{e}}(\alpha^+,\beta) = -2\eta[c_1\chi(\alpha^+) - c_2\zeta(\alpha^+)],$$
$$\eta G_{1n}^{\mathrm{e}}(\alpha^+,\beta) = -[c_1\zeta(\alpha^+) + c_2\chi(\alpha^+)]. \tag{20}$$

The arbitrary constants c_1 and c_2 are found from the nontangential conditions at the edge $\alpha = -\xi$ of the shell. Thus, in the case of the free edge, we obtain

$$G_{1n}(-\xi) = G_{1n}^{\mathrm{e}}(-\xi, c_1, c_2) + G_{1n}^{\mathrm{b}}(-\xi) = 0, \quad Q_{1n}(-\xi) = Q_{1n}^{\mathrm{e}}(-\xi, c_1, c_2) + Q_{1n}^{\mathrm{b}}(-\xi) = 0. \tag{21}$$

For the rigidly clamped edge, the procedure is similar. Note that, in (21), the quantities $G_{1n}^{\mathrm{b}}(-\xi)$ and

$Q^{\mathrm{b}}_{1n}(-\xi)$ are the amplitude values of the longitudinal bending moment and the shearing force of the basic state.

For the stress state for $\bar{n}+1 \leqslant n \leqslant n^*$, the resolving function is $\Phi^{\mathrm{s}}_n(\alpha)$, which is a solution of Eq. (8), where $\bar{\mathcal{L}}$ takes the form

$$\bar{\mathcal{L}} = \frac{d^8}{d\alpha^8} - a_{6,2}\tilde{n}^2\frac{d^6}{d\alpha^6} + a_{4,4}\tilde{n}^4\frac{d^4}{d\alpha^4} - a_{2,6}\tilde{n}^6\frac{d^2}{d\alpha^2} + \lambda^2\tilde{n}^8 + (1-\nu_1\nu_2)\,c^{-2}\lambda\frac{d^4}{d\alpha^4}. \tag{22}$$

The function $\Phi^{\mathrm{s}}_n(\alpha)$ should satisfy the conditions at infinity and at the edge $\alpha = -\xi$ of the shell:

$$\Phi^{\mathrm{s}}_n(\alpha) = c_1\exp(-r_{1n}\alpha)\sin s_{1n}\alpha + c_2\exp(-r_{1n}\alpha)\cos s_{1n}\alpha$$
$$+ c_3\exp(-r_{2n}\alpha)\sin s_{2n}\alpha + c_4\exp(-r_{2n}\alpha)\cos s_{2n}\alpha + \hat{\Phi}_n(\alpha). \tag{23}$$

Here r_{1n}, s_{1n}, r_{2n}, and s_{2n} are the real and imaginary parts of eight complex roots $æ_{1-4} = \pm r_{1n} \pm is_{1n}$ and $æ_{5-8} = \pm r_{2n} \pm is_{2n}$ of the following characteristic equation for the differential equation (8) with the operator (22) instead of the operator (9):

$$æ^8 - a_{6,2}\tilde{n}^2æ^6 + [a_{4,4}\tilde{n}^4 + (1-\nu_1\nu_2)\lambda/c^2]æ^4 - a_{2,6}\tilde{n}^6æ^2 + \lambda^2\tilde{n}^8 = 0.$$

The particular solution $\hat{\Phi}^{\mathrm{s}}_n(\alpha)$ can be determined by applying a Fourier transform to the differential equation (8) with the operator (22):

$$\hat{\Phi}^{\mathrm{s}}_n(\alpha) = \frac{1}{2\pi c^2}(\alpha_{2t} + \nu_1\alpha_{1t})\,Rt_0\theta_n\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{-i\omega\alpha}}{\mathcal{L}(\omega,n)}\,d\omega\int\limits_{-\infty}^{+\infty}\theta(\alpha)\mathrm{e}^{-i\omega\alpha}\,d\alpha,$$

$$\mathcal{L}(\omega,n) = \omega^8 + a_{6,2}\tilde{n}^2\omega^6 + a_{4,4}\tilde{n}^4\omega^4 + a_{2,6}\tilde{n}^6\omega^2 + \lambda^2\tilde{n}^8 + (1-\nu_1\nu_2)\,c^{-2}\lambda\omega^4.$$

The arbitrary constants c_1, c_2, c_3, and c_4 in (23) are found from the boundary conditions (10)–(13) at the free or rigidly clamped edge.

For the resolving function $\Phi^{\mathrm{t}}_n(\alpha) = \varphi_n(\alpha)$, according to (5) and (7), we obtain the ordinary differential equation

$$\left[\frac{d^4}{d\alpha^4} - \left(\frac{\lambda - \nu_2^2}{\mu_1} - 2\nu_2\right)\tilde{n}^2\frac{d^2}{d\alpha^2} + \lambda\tilde{n}^4\right]\Phi^{\mathrm{t}}_n(\alpha) = (\alpha_{2t} + \nu_1\alpha_{1t})\,Rt_0\theta_n\theta(\alpha),$$

which, after transformation of the coefficients with allowance for the relation for the shear modulus $G = (1/2)(E_1E_2)^{1/2}[1 + (\nu_1\nu_2)^{1/2}]^{-1}$, takes the form

$$\left(\frac{d^2}{d\alpha^2} - \lambda^{1/2}\tilde{n}^2\right)^2\Phi^{\mathrm{t}}_n(\alpha) = (\alpha_{2t} + \nu_1\alpha_{1t})\,Rt_0\theta_n\theta(\alpha). \tag{24}$$

With allowance for the conditions at infinity, the solution of Eq. (24) is

$$\Phi^{\mathrm{t}}_n(\alpha) = (c^{\mathrm{t}}_1 + c^{\mathrm{t}}_2\lambda^{1/4}\tilde{n}\alpha)\exp(-\lambda^{1/4}\tilde{n}\alpha) + \frac{\alpha_{2t} + \nu_1\alpha_{1t}}{2\pi}\,\theta_nRt_0\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{-i\omega\alpha}}{(\omega^2 + \lambda^{1/2}\tilde{n}^2)^2}\,d\omega\int\limits_{-\infty}^{+\infty}\theta(\alpha)\mathrm{e}^{i\omega\alpha}\,d\alpha. \tag{25}$$

The arbitrary constants in (25) are found from the requirement of satisfaction of the tangential boundary conditions (10) or (12) depending on whether the free edge or the clamped shell edge is considered.

Now, the resolving functions for each elementary state are constructed on the basis of the approximate equations (17), (19), (23), and (25), and it is not hard to obtain expressions for the desired displacements, forces, and bending moments. To do this, it is necessary to use relations that connect these factors with the resolving functions in semimomentless theory, in edge-effect theory, and in the cases of a tangential state and a stress state with high variability. Full expressions for the desired factors are obtained by summing the above-mentioned solutions:

$$u(\alpha,\beta) \approx u^{\mathrm{b}}(\alpha,\beta) + u^{\mathrm{s}}(\alpha,\beta) + u^{\mathrm{t}}(\alpha,\beta), \quad v(\alpha,\beta) \approx v^{\mathrm{b}}(\alpha,\beta) + v^{\mathrm{s}}(\alpha,\beta) + v^{\mathrm{t}}(\alpha,\beta),$$
$$w(\alpha,\beta) \approx w^{\mathrm{b}}(\alpha,\beta) + w^{\mathrm{e}}(\alpha,\beta) + w^{\mathrm{s}}(\alpha,\beta),$$

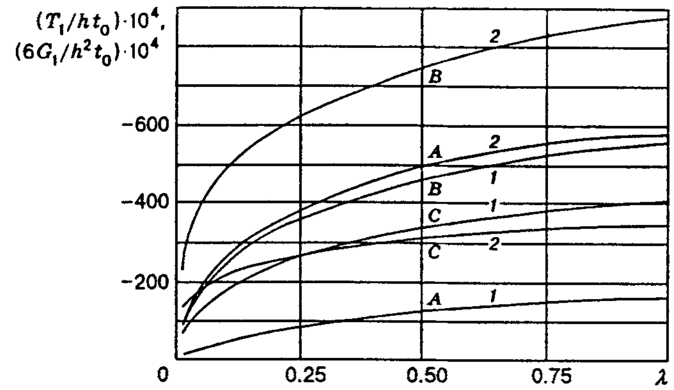

Fig. 1

$$
\begin{aligned}
T_1(\alpha,\beta) &\approx T_1^{\mathrm{b}}(\alpha,\beta) + T_1^{\mathrm{s}}(\alpha,\beta) + T_1^{\mathrm{t}}(\alpha,\beta) - T_{1t}(\alpha,\beta),\\
T_2(\alpha,\beta) &\approx T_2^{\mathrm{e}}(\alpha,\beta) + T_2^{\mathrm{s}}(\alpha,\beta) + T_2^{\mathrm{t}}(\alpha,\beta) - T_{2t}(\alpha,\beta),\\
G_1(\alpha,\beta) &\approx \nu_2 G_2^{\mathrm{b}}(\alpha,\beta) + G_1^{\mathrm{e}}(\alpha,\beta) + G_1^{\mathrm{s}}(\alpha,\beta),\\
G_2(\alpha,\beta) &\approx G_2^{\mathrm{b}}(\alpha,\beta) + \nu_2 G_1^{\mathrm{e}}(\alpha,\beta) + G_2^{\mathrm{s}}(\alpha,\beta),\\
T_{1t}(\alpha,\beta) &= E_1 h(1-\nu_1\nu_2)^{-1}(\alpha_{1t} + \nu_2\alpha_{2t})t(\alpha,\beta),\\
T_{2t}(\alpha,\beta) &= E_2 h(1-\nu_1\nu_2)^{-1}(\alpha_{2t} + \nu_1\alpha_{1t})t(\alpha,\beta).
\end{aligned} \tag{26}
$$

The resulting solution gives practically exact results at much less expenditures than with the use of the equations of general theory (1). For further simplifications, in (26), we omit terms with the superscript s, which refers to the solution based on Eq. (4), and expand the region of application of solutions for the basis state and the edge effect to $n = n^*$, which corresponds to the concept of the third MAS.

Thus far it has been assumed that, in the shell with a free or clamped edge a temperature field occurs in the form of arbitrarily shaped hot spots located uniformly along one contour, i.e., it is assumed that the temperature is represented in the form (6). In what follows, we shall study a semi-infinite shell with a free edge which is exposed to a piecewise-constant temperature field. This temperature distribution involves great computational difficulties, but, simultaneously, it allows one to examine the influence of various parameter of the shell, of the heated region, and of the material on the stress–strain state of the shell using this particular example. For the piecewise-constant temperature field, in the solutions written above one should set $\theta(\alpha) = 1$, $(|\alpha| \leqslant \alpha_0)$, $\theta(\alpha) = 0$ $(\alpha > \alpha_0)$, $\theta_n = k\beta_0/\pi$ $(n = 0)$, and $\theta_n = (2/\pi n)\sin kn\beta_0$ $(n = 1, 2, 3, \ldots)$.

We assume that the heated region is rectangular. Solutions for circular, elliptical, etc., regions can be obtained in a similar manner. Goodier [6] was apparently the first who considered these issues for the plane elastic problem. The solution obtained was used to analyze the influence of various parameters of the physically orthotropic material of the shell on the value and character of a stress–strain state. The subject of investigation was a shell with a free edge and with the parameter $h/R = 1/100$ in the presence in it of two square heated zones (hot spots) $0.25R \times 0.25R$ $(\alpha_0 = \beta_0 = 0.125)$

We first consider the results for an infinitely long shell $(\xi \to \infty)$. The dependence of the longitudinal force $T_1(0,0)$ (curves 1) and the longitudinal bending moment $G_1(0,0)$ (curves 2) on the parameter λ for various combinations of α_{1t} and α_{2t} is shown in Fig. 1, where curves A, B, and C correspond to the following values of α_{1t} and α_{2t}: $0.1\cdot 10^{-6}$ and $10\cdot 10^{-6}$ 1/°C, $10\cdot 10^{-6}$ and $10\cdot 10^{-6}$ 1/°C, and $10\cdot 10^{-6}$ and $0.1\cdot 10^{-6}$ 1/°C. The elastic moduli E_1 and E_2 were varied so that the orthotropy parameter λ varied within the range 0.01–1.0.

Similar information is given in Fig. 2 for the force $T_2(0,0)$ and the bending moment $G_2(0,0)$ (curves 1 and 2). The remaining notation is the same as in Fig. 1. Information on the stress state in the heated zone is readily illustrated by diagrams "longitudinal stress–circumferential stress," one of which is shown in Fig. 3 for $\alpha_{1t} = \alpha_{2t} = 10 \cdot 10^{-6}$ 1/°C at the center of the heated region. The left branch of the diagram (curve 1)

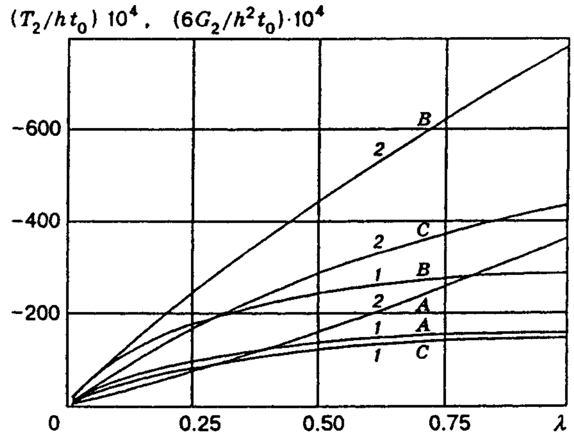

Fig. 2

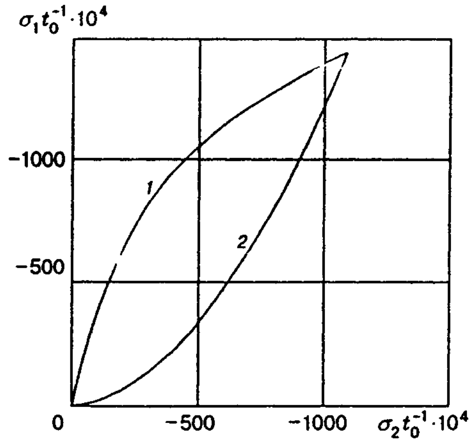

Fig. 3

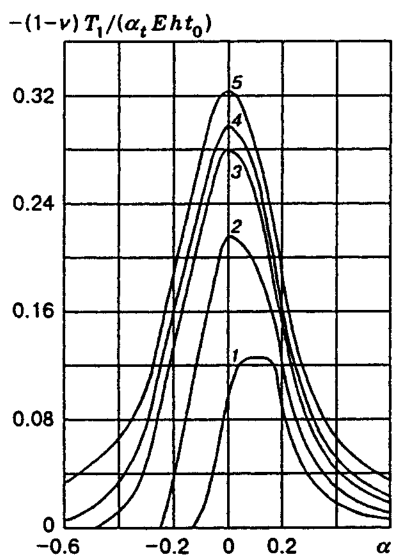

Fig. 4

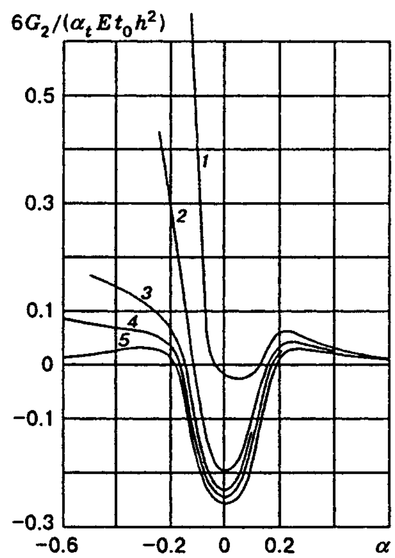

Fig. 5

corresponds to orthotropy variants of the material for which $E_1 = 2 \cdot 10^4$, and E_2 increases from the lower point (the coordinate origin) to the upper point so that the orthotropy parameter λ varies from 0.01 at the lower point to 1 at the upper point, where the left and right branches merge. The right branch of the diagram (curve 2) corresponds to orthotropy variants for which $E_2 = 2 \cdot 10^4$, and the elastic modulus E_1 decreases from the upper to the lower point. In this case, values of the orthotropy parameter λ vary from 1.0 at the upper point to 0.01 at the lower point. Thus, the upper point of the diagram corresponds to an isotropic material. At this point, the thermoelastic stresses are maximal. The left and right branches of the diagram envelop a family of curves that correspond to arbitrary values of the coefficients of linear expansion α_{1t} and α_{2t} and the elastic moduli E_1 and E_2 of the shell material. This means that the diagram gives maximum possible stresses at the center of the heated region. These diagrams appear to be of interest in designing materials for various high-temperature installations.

We consider the thermoelastic interaction of hot spots with the free edge of a shell with the parameter $h/R = 1/100$. Two heated regions $0.25R \times 0.25R$ ($\alpha_0 = \beta_0 = 0.125$) are located at a distance ξR from the free edge. The longitudinal force and the circular bending moment calculated using the third MAS are given in Figs. 4 and 5, respectively, where curves 1–5 correspond to $\xi = 0.126$, 0.25, 0.50, 0.75, and ∞. Note that curve 1 describes the behavior of the longitudinal force and the bending moment when the hot spots are adjacent to the free edge (the "cold" lintel separating the edge and the spot is equal to $0.001R$) As the hot spots approach the edge, the stress state changes: the bending stresses, which were maximal at the center of the heated region, become maximal at the shell edge and change sign. The largest value of the bending stress for the heating at the edge is more than twice as high as the largest stress for the case where the spots are well off the edge $\xi \to \infty$. This corresponds to the mechanical-mathematical model of an infinitely thin shell. In contrast, the longitudinal force decreases when the heated regions approach the free edge. When the spots are adjacent to the edge, it becomes minimal. Note that, according to the boundary condition (10), the longitudinal force at the free edge vanishes, and this can be easily verified by curves 1–3 in Fig. 4. In Figs. 4 and 5, it is assumed that $\lambda = 1$, $\nu_1 = \nu_2 = \nu$, and $\alpha_{1t} = \alpha_{2t} = \alpha_t$.

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